

Mapping the geometry of the E_6 group

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Abstract

In this paper we present a construction for the compact form of the exceptional Lie group E_6 by exponentiating the corresponding Lie algebra \mathfrak{e}_6 , which we realize as the the sum of \mathfrak{f}_4 , the derivations of the exceptional Jordan algebra J_3 of dimension 3 with octonionic entries, and the right multiplication by the elements of J_3 with vanishing trace. Our parametrization is a generalization of the Euler angles for $SU(2)$ and it is based on the fibration of E_6 via a F_4 subgroup as the fiber. It makes use of a similar construction we have performed in a previous article for F_4 . An interesting first application of these results lies in the fact that we are able to determine an explicit expression for the Haar invariant measure on the E_6 group manifold.

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1 Introduction.

The Standard Model (SM) provides a very good description of elementary particle physics. However, despite its success there are reasons to go beyond it: for example the recent discovery of neutrino oscillations, the fine tuning of the mixing matrices, the hierarchy problem, the difficulty in including gravity and so on.

A starting point could be the fact that the renormalization flow of the coupling constants suggests the unification of gauge interactions at energies of the order of 10^{15} GeV, which can be improved (fine tuned) by supersymmetry. It is natural to expect the gauge group G of the GUT theory to be a simple group. Obviously, at low energies the GUT model must reproduce the SM physics, so that not only G should contain $SU(3)_c \times SU(2)_L \times U(1)_Y$ (the SM gauge group), but it should also predict the correct spectra after spontaneous symmetry breaking. The increasing accuracy in the analysis of particles spectra imposes even more restrictions in the possible choices for G . E.g., it is already known that the current estimate for the lower bound of the proton lifetime rules out some of the GUT models candidates. Recently, the particular structure of the neutrino mixing matrix seems to suggest that a good candidate for a GUT could be based on the semidirect product between the exceptional group E_6^4 and the discrete group S_4 [3, 6].

In general, while the local properties of the group G , which for a Lie group are all encoded in the corresponding Lie algebra, are enough to perform a perturbative analysis, in order to obtain non perturbative results, such as instantonic calculations or lattice simulations, a knowledge of the entire group structure is required, in particular of the invariant measure on the group in a suitable global parametrization. There are many ways to give such an explicit expression for the Haar measure on a Lie group. However for large dimensions it becomes quite hard to find a realization, which is able at the same time to provide both a reasonably simple form for the measure and an explicit determination for the range of the angles. In this paper we solve this problem for the exceptional Lie group E_6 , by constructing a generalization of the Euler parametrization, with a technique we have introduced in [2] and fully developed in [1]. In section 2 we explain how the \mathfrak{e}_6 algebra can be represented using a theorem due to Chevalley and Schafer. The construction of the group and the determination of the corresponding Haar measure is made in section 3.

2 The construction of the \mathfrak{e}_6 algebra.

Our starting point for the construction of the exceptional algebra \mathfrak{e}_6 is a Theorem due to Chevalley and Schafer [5] which we rewrite here for convenience

Theorem 2.1 *The exceptional simple Lie algebra \mathfrak{f}_4 of dimension 52 and rank 4 over K is the derivation algebra \mathfrak{D} of the exceptional Jordan algebra \mathfrak{J} of dimension 27 over K . The exceptional simple Lie algebra \mathfrak{e}_6 of dimension 78 and rank 6 over K is the Lie algebra*

$$\mathfrak{D} + \{R_Y\} , \quad \text{Tr} Y = 0 , \quad (2.1)$$

spanned by the derivations of \mathfrak{J} and the right multiplications of elements Y of trace 0.

We will refer to [8] for the notations. Then the exceptional Jordan algebra is the algebra J_3 . For our purposes K will be \mathbb{R} or \mathbb{C} . The right multiplication is $R_Y(X) = Y \circ X$ and the trace is the

sum of diagonal elements. The exceptional Jordan algebra is 27 dimensional, just like the principal fundamental representation of \mathfrak{e}_6 .

In our previous paper [1] we determined the algebra of derivations \mathfrak{D} and used it to obtain an irreducible representation of the exceptional algebra \mathfrak{f}_4 . To obtain a representation of \mathfrak{e}_6 we only need to determine the matrix representation of R_Y . This is a very simple task and we solved it in \mathbb{R}^{27} with the product inherited from J_3 by means of the linear isomorphism

$$\Phi : J_3 \longrightarrow \mathbb{R}^{27}, \quad A \mapsto \Phi(A),$$

$$\Phi(A) := \begin{pmatrix} a_1 \\ \rho(o_1) \\ \rho(o_2) \\ a_2 \\ \rho(o_3) \\ a_3 \end{pmatrix}, \quad (2.2)$$

where A is the Jordan matrix

$$A = \begin{pmatrix} a_1 & o_1 & o_2 \\ o_1^* & a_2 & o_3 \\ o_2^* & o_3^* & a_3 \end{pmatrix}, \quad (2.3)$$

and ρ is the linear isomorphism between the octonions \mathbb{O} and \mathbb{R}^8 given by¹

$$\rho : \mathbb{O} \longrightarrow \mathbb{R}, \quad o = o^0 + \sum_{i=1}^7 o^i i_i \mapsto \rho(o),$$

$$\rho(o) := \begin{pmatrix} o^0 \\ o^1 \\ o^2 \\ o^3 \\ o^4 \\ o^5 \\ o^6 \\ o^7 \end{pmatrix}. \quad (2.4)$$

After choosing a 26 dimensional base of traceless Jordan matrices we used the Mathematica program in App.A to find the matrices which complete the base for \mathfrak{f}_4 to a base for the whole \mathfrak{e}_6 algebra. In particular if we work in the real case, we obtain the split form of \mathfrak{e}_6 , with signature (52, 26). However, multiplying the 26 added generators by i we find that the algebra remains real, but the Killing form becomes the compact one.

¹See [1] for more details.

3 The construction of the group E_6

3.1 The generalized Euler parametrization for E_6

To give an Euler parametrization for E_6 we start by choosing its maximal subgroup. It is $H = F_4$, the group generated by the first 52 matrices. Let \mathcal{P} be the linear complement of \mathfrak{f}_4 in \mathfrak{e}_6 . We then search for a minimal linear subset V of \mathcal{P} , which, under action of $Ad(F_4)$, generates the whole \mathcal{P} . Looking at the structure constants we see that V can be chosen as the linear space generated by c_{53}, c_{70} . Note that they commute.

If we then write the general element g of E_6 in the form

$$g = \exp(\tilde{h}) \exp(v) \exp(h) , \quad h, \tilde{h} \in \mathfrak{f}_4 , \quad v \in V , \quad (3.1)$$

we have a redundancy of dimension 28. As argued in [1] we expect to find a 28-dimensional subgroup of F_4 , which, acting by adjunction, defines an automorphism of V . This can be done by noticing that V commutes with the first 28 matrices c_i , $i = 1, \dots, 28$, which generate an $SO(8)$ subgroup of F_4 .

We found convenient to introduce the change of base

$$\tilde{c}_{53} := \frac{1}{2}c_{53} + \frac{\sqrt{3}}{2}c_{70} , \quad (3.2)$$

$$\tilde{c}_{70} := -\frac{\sqrt{3}}{2}c_{53} + \frac{1}{2}c_{70} . \quad (3.3)$$

Thus we have

$$g[x_1, \dots, x_{78}] = B[x_1, \dots, x_{24}] e^{x_{25}\tilde{c}_{53}} e^{x_{26}\tilde{c}_{70}} F_4[x_{27}, \dots, x_{78}] ,$$

where $B = F_4/SO(8)$. We chose for F_4 the Euler parametrization given in [1] so that we found for B

$$B[x_1, \dots, x_{24}] = B_{F_4}[x_1, \dots, x_{16}] B_9[x_{17}, \dots, x_{23}] e^{x_{24}c_{45}} , \quad (3.4)$$

where

$$B_{F_4}[x_1, \dots, x_{15}] = B_9[x_1, \dots, x_7] e^{x_8c_{45}} B_8[x_9, \dots, x_{14}] e^{x_{15}\tilde{c}_{30}} e^{x_{16}c_{22}} , \quad (3.5)$$

$$B_9[x_1, \dots, x_7] = e^{x_1\tilde{c}_3} e^{x_2\tilde{c}_{16}} e^{x_3\tilde{c}_{15}} e^{x_4\tilde{c}_{35}} e^{x_5\tilde{c}_5} e^{x_6\tilde{c}_1} e^{x_7\tilde{c}_{30}} , \quad (3.6)$$

$$B_8[x_1, \dots, x_6] = e^{x_1\tilde{c}_3} e^{x_2\tilde{c}_{16}} e^{x_3\tilde{c}_{15}} e^{x_4\tilde{c}_{35}} e^{x_5\tilde{c}_5} e^{x_6\tilde{c}_1} , \quad (3.7)$$

and the tilded matrices are the ones introduced in [1].

3.2 Determination of the range for the parameters.

To determine the range of the parameters we will use the topological method we have developed in [2]. Let us first determine the volume of E_6 by means of the Macdonald formula.

3.2.1 The volume of E_6 .

The rational homology of the exceptional Lie group E_6 is that of a product of odd dimensional spheres [10], $H_*(E_6) = H_*(\prod_{i=1}^6 S^{d_i})$, with ([4])

$$d_1 = 3, d_2 = 9, d_3 = 11, d_4 = 15, d_5 = 17, d_6 = 23. \quad (3.8)$$

The simple roots of E_6 are

$$r_1 = L_1 + L_2, \quad (3.9)$$

$$r_2 = L_2 - L_1, \quad (3.10)$$

$$r_3 = L_3 - L_2, \quad (3.11)$$

$$r_4 = L_4 - L_3, \quad (3.12)$$

$$r_5 = L_5 - L_4, \quad (3.13)$$

$$r_6 = \frac{L_1 - L_2 - L_3 - L_4 - L_5 + \sqrt{3}L_6}{2}, \quad (3.14)$$

where L_i , $i = 1, \dots, 6$ is an orthonormal base for the Cartan algebra. The volume of the fundamental region is then

$$Vol(f_{E_6}) = \frac{2}{L}. \quad (3.15)$$

Indeed we computed the 36 positive roots, all having length $\sqrt{2}$. They have exactly the structure given in [8], with $L_i = e_i$, the canonical base of \mathbb{R}^6 . The Macdonald formula [9], [11] gives for the volume of the compact form of E_6

$$Vol(E_6) = \frac{\sqrt{3} \cdot 2^{17} \cdot \pi^{42}}{3^{10} \cdot 5^5 \cdot 7^3 \cdot 11}. \quad (3.16)$$

3.2.2 The invariant measure on E_6 .

With the chosen generalized Euler parametrization, the invariant measure on E_6 decomposes into the product of the measure of F_4 and the one on $M = E_6/F_4$. The invariant measure on F_4 was computed in [1] so that we need to compute here only the induced measure on M .

Using the notation of [1], let us define

$$J_M := \pi_{\mathcal{P}}(e^{-x_{26}\tilde{c}_{70}} e^{-x_{25}\tilde{c}_{53}} B[x_1, \dots, x_{24}]^{-1} d(B[x_1, \dots, x_{24}] e^{x_{25}\tilde{c}_{53}} e^{x_{26}\tilde{c}_{70}})) , \quad (3.17)$$

where $\pi_{\mathcal{P}}$ is the projection on the subspace generated by c_j , $j = 53, \dots, 78$. The metric induced on M by the bi-invariant metric on E_6 is then

$$ds_M^2 = -\frac{1}{6} \text{Trace}(J_M \otimes J_M), \quad (3.18)$$

and the invariant measure on E_6 is then

$$d\mu_{E_6} = |\det(J_{Mi}^j)| d\mu_{F_4} \prod_{l=1}^{26} dx_l, \quad (3.19)$$

where J_{Mi}^j is the 26×26 matrix defined by

$$J_M = \sum_{i,j=1}^{26} J_{Mi}^j c_{ij} dx^i , \quad (3.20)$$

with c_{ij} a base $\{\tilde{c}_{53}, \tilde{c}_{70}, c_{54}, \dots, c_{69}, c_{71}, \dots, c_{78}\}$ of \mathcal{P} .

In order to compute $|\det(J_{Mi}^j)|$ it is convenient to introduce the notations

$$\omega[x, y, z] = e^{xc_{45}} e^{y\tilde{c}_{53}} e^{z\tilde{c}_{70}} , \quad (3.21)$$

$$J_\omega := \omega^{-1} d\omega , \quad (3.22)$$

$$J_9 := B_9^{-1} dB_9 , \quad (3.23)$$

$$J_{F_4} := B_{F_4}^{-1} dB_{F_4} , \quad (3.24)$$

so that

$$\begin{aligned} J_M[x_1, \dots, x_{26}] &= \omega[x_{24}, x_{25}, x_{26}]^{-1} B_9[x_{17}, \dots, x_{23}]^{-1} J_{F_4}[x_1, \dots, x_{16}] B_9[x_{17}, \dots, x_{23}] \omega[x_{24}, x_{25}, x_{26}] \\ &\quad + \omega[x_{24}, x_{25}, x_{26}]^{-1} J_9[x_{17}, \dots, x_{23}] \omega[x_{24}, x_{25}, x_{26}] + J_\omega[x_{24}, x_{25}, x_{26}] . \end{aligned} \quad (3.25)$$

Some remarks are in order now

1. the following relations are true

$$e^{-\alpha\tilde{c}_{53}} c_L e^{\alpha\tilde{c}_{53}} = \cos \alpha \ c_L + \sin \alpha \ c_{L+26} , \quad (3.26)$$

if $L = 45, \dots, 52$. Moreover \tilde{c}_{70} commutes with c_I , $I = 45, \dots, 52, 71, \dots, 78$ and with \tilde{c}_{53} ;

2. \tilde{c}_{53} and \tilde{c}_{70} commute with the $so(8)$ algebra generated by the matrices c_I , $I = 1, \dots, 21, 30, \dots, 36$;
3. the adjoint action of $e^{x_{24}c_{45}}$ on the above $so(8)$ algebra generates in addition the matrices c_J , $J = 46, \dots, 52$;
4. from $J_{F_4} \in \mathfrak{f}_4$ and $B_9 \subset F_4$, it follows that

$$\tilde{J}_{F_4} := B_9^{-1} J_{F_4} B_9 \in \mathfrak{f}_4 ;$$

5. the adjoint action of ω on the \mathfrak{f}_4 matrices generates all the remaining matrices of \mathfrak{e}_6 . In particular, the projection of $\omega^{-1} c_L \omega$, $L = 1, \dots, 52$ on c_J , $J = 54, \dots, 69$, is different from zero only if $L = 22, \dots, 29, 37, \dots, 44$;
6. the adjoint action of the $SO(8)$ group, corresponding to the above $so(8)$ algebra, gives a rotation both on the indexes $I = \{22, \dots, 29\}$ and $J = \{37, \dots, 44\}$. More precisely

$$SO(8)^{-1} c_I SO(8) = R_I^L c_L , \quad (3.27)$$

$$SO(8)^{-1} c_J SO(8) = \tilde{R}_J^K c_K , \quad (3.28)$$

where L runs from 22 to 29 and K runs from 37 to 44, and R, \tilde{R} are both orthogonal matrices. To verify these it suffices to note that

$$e^{-xc_A} c_I e^{xc_A} = \cos \frac{x}{2} c_I \pm \sin \frac{x}{2} c_{I_A} , \quad (3.29)$$

$$e^{-xc_A} c_J e^{xc_A} = \cos \frac{x}{2} c_J \pm \sin \frac{x}{2} c_{J_A} , \quad (3.30)$$

where $A = 1, \dots, 21, 30, \dots, 36$, $I, I_A \in \{22, \dots, 29\}$, $J, J_A \in \{37, \dots, 44\}$. In particular $\det(R \otimes \tilde{R}) = 1$.

From the first three points, we see that the matrix J_M takes the form

$$M = \begin{pmatrix} A & 0 & 0 \\ * & C & 0 \\ * & * & D \end{pmatrix} , \quad (3.31)$$

where on the rows we indicate the coefficients of dx_I with $I = 1, \dots, 26$ starting from the bottom, and on the columns the projections on \tilde{c}_A, c_L , following the order: $A = 53, 70$, $L = 71, \dots, 78, 54, \dots, 69$. The asterisks indicate the elements which do not contribute to the determinant:

$$\det J_M = \det A \det C \det D .$$

In particular, A is a 3×3 block obtained by projecting J_ω on $\tilde{c}_{53}, \tilde{c}_{70}, c_{71}$. Point 1 implies

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sin x_{25} \end{pmatrix} , \quad (3.32)$$

so that

$$\det A = \sin x_{25} . \quad (3.33)$$

The C block is a 7×7 matrix obtained by projecting $\omega^{-1} J_9 \omega$ on c_K , $K = 72, \dots, 78$. From point 1 it follows that it can be obtained by multiplying the projection of $e^{-x_{24} c_{45}} J_9 e^{x_{24} c_{45}}$ on c_L , $L = 46, \dots, 52$ by $\sin x_{25}$. If we call \tilde{C} the matrix corresponding to such a projection, we find $\det C = \sin^7 x_{25} \det \tilde{C}$. The determinant of \tilde{C} can be computed directly and gives

$$\det C = \sin x_{20} \cos x_{21} \cos x_{22} \sin^2 x_{22} \sin^2 x_{23} \cos^4 x_{23} \sin^7 x_{24} \sin^7 x_{25} . \quad (3.34)$$

The D block requires some further discussion. It is a 16×16 matrix obtained by projecting $\omega^{-1} \tilde{J}_{B_{F_4}} \omega$ on c_I , $I = 54, \dots, 69$. First, from points 4, 5 of our remarks we see that only the c_L with $L = 22, \dots, 29, 37, \dots, 44$, in \tilde{J}_{F_4} contribute to the determinant. Let us define the 16×16 matrix U with

$$U_A^B := -\frac{1}{6} \text{Tr} (\omega^{-1} c_A \omega c_B) , \quad (3.35)$$

$$A = 22, \dots, 29, 37, \dots, 44 , \quad (3.36)$$

$$B = 54, \dots, 69 . \quad (3.37)$$

Moreover let \tilde{D} be the matrix obtained by projecting J_{F_4} on c_L with $L = 22, \dots, 29, 37, \dots, 44$, and

$$Q := \begin{pmatrix} R & 0 \\ 0 & \tilde{R} \end{pmatrix}, \quad (3.38)$$

where R and \tilde{R} are the rotation matrices defined at point 6. Then we can deduce from points 4, 5 and 6 that

$$\det D = \det(U \cdot Q \cdot \tilde{D}) = \det U \det \tilde{D}. \quad (3.39)$$

The matrix U and its determinant are easily computed. Indeed, we have

$$\begin{aligned} \omega^{-1}c_{22}\omega &= \cos\left(\frac{x_{24}}{2}\right) \sin\left(\frac{\sqrt{3}}{2}x_{26} + \frac{x_{25}}{2}\right) c_{61} + \sin\left(\frac{x_{24}}{2}\right) \sin\left(\frac{\sqrt{3}}{2}x_{26} - \frac{x_{25}}{2}\right) c_{69} + \dots \\ \omega^{-1}c_{23}\omega &= -\cos\left(\frac{x_{24}}{2}\right) \sin\left(\frac{\sqrt{3}}{2}x_{26} + \frac{x_{25}}{2}\right) c_{57} - \sin\left(\frac{x_{24}}{2}\right) \sin\left(\frac{\sqrt{3}}{2}x_{26} - \frac{x_{25}}{2}\right) c_{65} + \dots \\ \omega^{-1}c_{24}\omega &= -\cos\left(\frac{x_{24}}{2}\right) \sin\left(\frac{\sqrt{3}}{2}x_{26} + \frac{x_{25}}{2}\right) c_{60} - \sin\left(\frac{x_{24}}{2}\right) \sin\left(\frac{\sqrt{3}}{2}x_{26} - \frac{x_{25}}{2}\right) c_{68} + \dots \\ \omega^{-1}c_{25}\omega &= \cos\left(\frac{x_{24}}{2}\right) \sin\left(\frac{\sqrt{3}}{2}x_{26} + \frac{x_{25}}{2}\right) c_{55} + \sin\left(\frac{x_{24}}{2}\right) \sin\left(\frac{\sqrt{3}}{2}x_{26} - \frac{x_{25}}{2}\right) c_{63} + \dots \\ \omega^{-1}c_{26}\omega &= -\cos\left(\frac{x_{24}}{2}\right) \sin\left(\frac{\sqrt{3}}{2}x_{26} + \frac{x_{25}}{2}\right) c_{59} - \sin\left(\frac{x_{24}}{2}\right) \sin\left(\frac{\sqrt{3}}{2}x_{26} - \frac{x_{25}}{2}\right) c_{67} + \dots \\ \omega^{-1}c_{27}\omega &= \cos\left(\frac{x_{24}}{2}\right) \sin\left(\frac{\sqrt{3}}{2}x_{26} + \frac{x_{25}}{2}\right) c_{58} + \sin\left(\frac{x_{24}}{2}\right) \sin\left(\frac{\sqrt{3}}{2}x_{26} - \frac{x_{25}}{2}\right) c_{66} + \dots \\ \omega^{-1}c_{28}\omega &= \cos\left(\frac{x_{24}}{2}\right) \sin\left(\frac{\sqrt{3}}{2}x_{26} + \frac{x_{25}}{2}\right) c_{56} + \sin\left(\frac{x_{24}}{2}\right) \sin\left(\frac{\sqrt{3}}{2}x_{26} - \frac{x_{25}}{2}\right) c_{64} + \dots \\ \omega^{-1}c_{29}\omega &= -\cos\left(\frac{x_{24}}{2}\right) \sin\left(\frac{\sqrt{3}}{2}x_{26} + \frac{x_{25}}{2}\right) c_{54} - \sin\left(\frac{x_{24}}{2}\right) \sin\left(\frac{\sqrt{3}}{2}x_{26} - \frac{x_{25}}{2}\right) c_{62} + \dots \\ \omega^{-1}c_{37}\omega &= -\sin\left(\frac{x_{24}}{2}\right) \sin\left(\frac{\sqrt{3}}{2}x_{26} + \frac{x_{25}}{2}\right) c_{54} + \cos\left(\frac{x_{24}}{2}\right) \sin\left(\frac{\sqrt{3}}{2}x_{26} - \frac{x_{25}}{2}\right) c_{62} + \dots \\ \omega^{-1}c_{38}\omega &= -\sin\left(\frac{x_{24}}{2}\right) \sin\left(\frac{\sqrt{3}}{2}x_{26} + \frac{x_{25}}{2}\right) c_{55} + \cos\left(\frac{x_{24}}{2}\right) \sin\left(\frac{\sqrt{3}}{2}x_{26} - \frac{x_{25}}{2}\right) c_{63} + \dots \\ \omega^{-1}c_{39}\omega &= -\sin\left(\frac{x_{24}}{2}\right) \sin\left(\frac{\sqrt{3}}{2}x_{26} + \frac{x_{25}}{2}\right) c_{56} + \cos\left(\frac{x_{24}}{2}\right) \sin\left(\frac{\sqrt{3}}{2}x_{26} - \frac{x_{25}}{2}\right) c_{64} + \dots \\ \omega^{-1}c_{40}\omega &= -\sin\left(\frac{x_{24}}{2}\right) \sin\left(\frac{\sqrt{3}}{2}x_{26} + \frac{x_{25}}{2}\right) c_{57} + \cos\left(\frac{x_{24}}{2}\right) \sin\left(\frac{\sqrt{3}}{2}x_{26} - \frac{x_{25}}{2}\right) c_{65} + \dots \\ \omega^{-1}c_{41}\omega &= -\sin\left(\frac{x_{24}}{2}\right) \sin\left(\frac{\sqrt{3}}{2}x_{26} + \frac{x_{25}}{2}\right) c_{58} + \cos\left(\frac{x_{24}}{2}\right) \sin\left(\frac{\sqrt{3}}{2}x_{26} - \frac{x_{25}}{2}\right) c_{66} + \dots \end{aligned}$$

$$\begin{aligned}
\omega^{-1}c_{42}\omega &= -\sin\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26} + \frac{x_{25}}{2}\right)c_{59} + \cos\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26} - \frac{x_{25}}{2}\right)c_{67} + \dots \\
\omega^{-1}c_{43}\omega &= -\sin\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26} + \frac{x_{25}}{2}\right)c_{60} + \cos\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26} - \frac{x_{25}}{2}\right)c_{68} + \dots \\
\omega^{-1}c_{44}\omega &= -\sin\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26} + \frac{x_{25}}{2}\right)c_{61} + \cos\left(\frac{x_{24}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}x_{26} - \frac{x_{25}}{2}\right)c_{69} + \dots
\end{aligned}$$

where ellipses stay for terms which vanish when projected on c_B , $B = 54, \dots, 61, 70, \dots, 78$. From these it follows

$$\det U = \sin^8\left(\frac{\sqrt{3}}{2}x_{26} + \frac{x_{25}}{2}\right)\sin^8\left(\frac{\sqrt{3}}{2}x_{26} - \frac{x_{25}}{2}\right). \quad (3.40)$$

At this point we need to compute $\det \tilde{D}$. This can be done by noticing that J_{F_4} coincides exactly with the current J_M of F_4 in [1]. We then find

$$\begin{aligned}
\det \tilde{D} &= 2^7 \sin^{15} \frac{x_{16}}{2} \cos^7 \frac{x_{16}}{2} \sin x_4 \cos x_5 \cos x_6 \sin^2 x_6 \cos^4 x_7 \sin^2 x_7 \sin^7 x_8 \cdot \\
&\quad \cdot \sin x_{12} \cos x_{13} \cos x_{14} \sin^2 x_{14} \cos^2 x_{15} \sin^4 x_{15}. \quad (3.41)
\end{aligned}$$

We can finally write down the invariant measure on the base space:

$$\begin{aligned}
d\mu_M &= 2^7 \sin x_4 \cos x_5 \cos x_6 \sin^2 x_6 \cos^4 x_7 \sin^2 x_7 \sin^7 x_8 \cdot \\
&\quad \cdot \sin x_{12} \cos x_{13} \cos x_{14} \sin^2 x_{14} \cos^2 x_{15} \sin^4 x_{15} \sin^{15} \frac{x_{16}}{2} \cos^7 \frac{x_{16}}{2} \cdot \\
&\quad \cdot \sin x_{20} \cos x_{21} \cos x_{22} \sin^2 x_{22} \sin^2 x_{23} \cos^4 x_{23} \sin^7 x_{24} \cdot \\
&\quad \cdot \sin^8 x_{25} \sin^8 \left(\frac{\sqrt{3}}{2}x_{26} + \frac{x_{25}}{2} \right) \sin^8 \left(\frac{\sqrt{3}}{2}x_{26} - \frac{x_{25}}{2} \right). \quad (3.42)
\end{aligned}$$

Note that the periods of the variables are 4π so that one should take the range $x_i = [0, 4\pi]$ for $i = 1, 2, 3, \dots, 9, 10, 11, 17, 18, 19$. However, as in [1], it is easy to show directly from the parametrization that they can all be restricted to $[0, 2\pi]$. The range of x_i is then

$$\begin{aligned}
x_1 &\in [0, 2\pi], & x_2 &\in [0, 2\pi], & x_3 &\in [0, 2\pi], & x_4 &\in [0, \pi], \\
x_5 &\in [-\frac{\pi}{2}, \frac{\pi}{2}], & x_6 &\in [0, \frac{\pi}{2}], & x_7 &\in [0, \frac{\pi}{2}], & x_8 &\in [0, \pi], \\
x_9 &\in [0, 2\pi], & x_{10} &\in [0, 2\pi], & x_{11} &\in [0, 2\pi], & x_{12} &\in [0, \pi], \\
x_{13} &\in [-\frac{\pi}{2}, \frac{\pi}{2}], & x_{14} &\in [0, \frac{\pi}{2}], & x_{15} &\in [0, \frac{\pi}{2}], & x_{16} &\in [0, \pi], \\
x_{17} &\in [0, 2\pi], & x_{18} &\in [0, 2\pi], & x_{19} &\in [0, 2\pi], & x_{20} &\in [0, \pi], \\
x_{21} &\in [-\frac{\pi}{2}, \frac{\pi}{2}], & x_{22} &\in [0, \frac{\pi}{2}], & x_{23} &\in [0, \frac{\pi}{2}], & x_{24} &\in [0, \pi], \\
x_{25} &\in [0, \frac{\pi}{2}], & -\frac{x_{25}}{\sqrt{3}} &\leq x_{26} \leq \frac{x_{25}}{\sqrt{3}}. \quad (3.43)
\end{aligned}$$

The remaining parameters x_j , $j = 27, \dots, 78$, will run over the range for F_4 , as given in [1]. The volume of the whole closed cycle V so obtained is then

$$Vol(V) = Vol(F_4) \int_R d\mu_M = \frac{\sqrt{3} \cdot 2^{17} \cdot \pi^{42}}{3^{10} \cdot 5^5 \cdot 7^3 \cdot 11}, \quad (3.44)$$

where R is the range of parameters x_i , $i = 1, \dots, 26$. This is the volume of E_6 , so that we cover the group exactly once.²

4 Conclusions.

In this paper we have performed an explicit construction for the compact form of the simple Lie group E_6 . This is particularly interesting, because recently it has been argued that this group could be the most promising for unification in GUT theories [3, 6]. To parameterize the group we have used the generalized Euler angles method, a technique we introduced in [2, 1] to give the most simple expression for the invariant measure on the group, while at the same time still being able to provide an explicit expression for the range of the parameters. Both these requirements are necessary in order to minimize the computation power needed for computer simulations, for example, of lattice models.

Our results can be easily extended to the GUT group $E_6^4 \rtimes S_4$. Also, a modified Euler parametrization could be applied to evidentiate the subgroups related to the correct symmetry breaking, see [3]. Finally, our parametrization could be used for a straightforward geometrical analysis of the exceptional Lie group E_6 and its quotients.

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²Obviously there is a subset of vanishing measure which is multiply covered.

A The ϵ_6 matrices.

The matrices we found using Mathematica, and orthonormalized with respect to the scalar product $\langle a, b \rangle := -\frac{1}{6} \text{Trace}(ab)$, was computed by means of the followings programs.

The Mathematica program.

```

%%%% the octonionic products
QQ[1, 1] = e[1];    QQ[1, 2] = e[2];    QQ[1, 3] = e[3];
QQ[1, 4] = e[4];    QQ[1, 5] = e[5];    QQ[1, 6] = e[6];
QQ[1, 7] = e[7];    QQ[1, 8] = e[8];    QQ[2, 1] = e[2];
QQ[2, 2] = -e[1];   QQ[2, 3] = e[5];    QQ[2, 4] = e[8];
QQ[2, 5] = -e[3];   QQ[2, 6] = e[7];    QQ[2, 7] = -e[6];
QQ[2, 8] = -e[4];   QQ[3, 1] = e[3];    QQ[3, 2] = -e[5];
QQ[3, 3] = -e[1];   QQ[3, 4] = e[6];    QQ[3, 5] = e[2];
QQ[3, 6] = -e[4];   QQ[3, 7] = e[8];    QQ[3, 8] = -e[7];
QQ[4, 1] = e[4];    QQ[4, 2] = -e[8];   QQ[4, 3] = -e[6];
QQ[4, 4] = -e[1];   QQ[4, 5] = e[7];    QQ[4, 6] = e[3];
QQ[4, 7] = -e[5];   QQ[4, 8] = e[2];    QQ[5, 1] = e[5];
QQ[5, 2] = e[3];    QQ[5, 3] = -e[2];   QQ[5, 4] = -e[7];
QQ[5, 5] = -e[1];   QQ[5, 6] = e[8];    QQ[5, 7] = e[4];
QQ[5, 8] = -e[6];   QQ[6, 1] = e[6];    QQ[6, 2] = -e[7];
QQ[6, 3] = e[4];    QQ[6, 4] = -e[3];   QQ[6, 5] = -e[8];
QQ[6, 6] = -e[1];   QQ[6, 7] = e[2];    QQ[6, 8] = e[5];
QQ[7, 1] = e[7];    QQ[7, 2] = e[6];    QQ[7, 3] = -e[8];
QQ[7, 4] = e[5];    QQ[7, 5] = -e[4];   QQ[7, 6] = -e[2];
QQ[7, 7] = -e[1];   QQ[7, 8] = e[3];    QQ[8, 1] = e[8];
QQ[8, 2] = e[4];    QQ[8, 3] = e[7];    QQ[8, 4] = -e[2];
QQ[8, 5] = e[6];    QQ[8, 6] = -e[5];   QQ[8, 7] = -e[3];
QQ[8, 8] = -e[1];

%%%% the Jordan algebra product
Qm[x_, y_] := Sum[Sum[x[[i]]y[[j]]QQ[i, j], {i, 8}], {j, 8}]
QP[x_, y_] := {Coefficient[Qm[x, y], e[1]], Coefficient[Qm[x, y], e[2]],
  Coefficient[Qm[x, y], e[3]], Coefficient[Qm[x, y], e[4]], Coefficient[Qm[x, y], e[5]],
  Coefficient[Qm[x, y], e[6]], Coefficient[Qm[x, y], e[7]], Coefficient[Qm[x, y], e[8]]}

```

```

OctP[a_, b_] := {{Sum[QP[Part[Part[a, 1], i], Part[Part[b, i], 1]], {i, 3}] ,
Sum[QP[Part[Part[a, 1], i], Part[Part[b, i], 2]], {i, 3}],
Sum[QP[Part[Part[a, 1], i], Part[Part[b, i], 3]], {i, 3}]} ,
{Sum[QP[Part[Part[a, 2], i], Part[Part[b, i], 1]], {i, 3}],
Sum[QP[Part[Part[a, 2], i], Part[Part[b, i], 2]], {i, 3}],
Sum[QP[Part[Part[a, 2], i], Part[Part[b, i], 3]], {i, 3}]} ,
{Sum[QP[Part[Part[a, 3], i], Part[Part[b, i], 1]], {i, 3}],
Sum[QP[Part[Part[a, 3], i], Part[Part[b, i], 2]], {i, 3}],
Sum[QP[Part[Part[a, 3], i], Part[Part[b, i], 3]], {i, 3}]}}
OctPS[a_, b_] := 1/2(OctP[a, b] + OctP[b, a])

```

%%% correspondence between the Jordan algebra and \mathbb{R}^{27}

```

FF[AA_] := {Part[{Part[Part[Part[AA, 1], 1], 1]}, 1],
Part[{Part[Part[Part[AA, 1], 2], 1]}, 1], Part[{Part[Part[Part[AA, 1], 2], 2]}, 1],
Part[{Part[Part[Part[AA, 1], 2], 3]}, 1], Part[{Part[Part[Part[AA, 1], 2], 4]}, 1],
Part[{Part[Part[Part[AA, 1], 2], 5]}, 1], Part[{Part[Part[Part[AA, 1], 2], 6]}, 1],
Part[{Part[Part[Part[AA, 1], 2], 7]}, 1], Part[{Part[Part[Part[AA, 1], 2], 8]}, 1],
Part[{Part[Part[Part[AA, 1], 3], 1]}, 1], Part[{Part[Part[Part[AA, 1], 3], 2]}, 1],
Part[{Part[Part[Part[AA, 1], 3], 3]}, 1], Part[{Part[Part[Part[AA, 1], 3], 4]}, 1],
Part[{Part[Part[Part[AA, 1], 3], 5]}, 1], Part[{Part[Part[Part[AA, 1], 3], 6]}, 1],
Part[{Part[Part[Part[AA, 1], 3], 7]}, 1], Part[{Part[Part[Part[AA, 1], 3], 8]}, 1],
Part[{Part[Part[Part[AA, 2], 2], 1]}, 1], Part[{Part[Part[Part[AA, 2], 3], 1]}, 1],
Part[{Part[Part[Part[AA, 2], 3], 2]}, 1], Part[{Part[Part[Part[AA, 2], 3], 3]}, 1],
Part[{Part[Part[Part[AA, 2], 3], 4]}, 1], Part[{Part[Part[Part[AA, 2], 3], 5]}, 1],
Part[{Part[Part[Part[AA, 2], 3], 6]}, 1], Part[{Part[Part[Part[AA, 2], 3], 7]}, 1],
Part[{Part[Part[Part[AA, 2], 3], 8]}, 1], Part[{Part[Part[Part[AA, 3], 3], 1]}, 1]}

```

```

FFi[vv_] :=
  {{{Part[vv, 1], 0, 0, 0, 0, 0, 0, 0, 0}, {Part[vv, 2], Part[vv, 3], Part[vv, 4], Part[vv, 5],
    Part[vv, 6], Part[vv, 7], Part[vv, 8], Part[vv, 9]}, {Part[vv, 10], Part[vv, 11],
    Part[vv, 12], Part[vv, 13], Part[vv, 14], Part[vv, 15], Part[vv, 16], Part[vv, 17]}},
  {{{Part[vv, 2], -Part[vv, 3], -Part[vv, 4], -Part[vv, 5], -Part[vv, 6],
    -Part[vv, 7], -Part[vv, 8], -Part[vv, 9]}, {Part[vv, 18], 0, 0, 0, 0, 0, 0, 0, 0},
  {Part[vv, 19], Part[vv, 20], Part[vv, 21], Part[vv, 22],
    Part[vv, 23], Part[vv, 24], Part[vv, 25], Part[vv, 26]}},

  {{{Part[vv, 10], -Part[vv, 11], -Part[vv, 12], -Part[vv, 13],
    -Part[vv, 14], -Part[vv, 15], -Part[vv, 16], -Part[vv, 17]},
  {Part[vv, 19], -Part[vv, 20], -Part[vv, 21], -Part[vv, 22], -Part[vv, 23],
    -Part[vv, 24], -Part[vv, 25], -Part[vv, 26]}, {Part[vv, 27], 0, 0, 0, 0, 0, 0, 0, 0}}}

```

%%% construction of the matrices

```

MT = { {{mt[1], 0, 0, 0, 0, 0, 0, 0, 0}, {mt[2], mt[3], mt[4], mt[5], mt[6], mt[7], mt[8],
  mt[9]}, {mt[10], mt[11], mt[12], mt[13], mt[14], mt[15], mt[16], mt[17]}},
  {{mt[2], -mt[3], -mt[4], -mt[5], -mt[6], -mt[7], -mt[8], -mt[9]}, {mt[18], 0, 0,
    0, 0, 0, 0, 0}, {mt[19], mt[20], mt[21], mt[22], mt[23], mt[24], mt[25], mt[26]}},
  {{mt[10], -mt[11], -mt[12], -mt[13], -mt[14], -mt[15], -mt[16], -mt[17]},
  {mt[19], -mt[20], -mt[21], -mt[22], -mt[23], -mt[24], -mt[25], -mt[26]},
  {-mt[1] - mt[18], 0, 0, 0, 0, 0, 0, 0, 0}}};

```

```

h = Array[hh, 27];
M = Array[mm, {27, 27}];
imm = FF[OctPS[MT, FFi[h]]];
Do[Do[mm[i, j] = Coefficient[Part[imm, i], Part[h, j]], {i, 27}], {j, 27}];
Do[econi[j] = D[M, mt[j]], {j, 26}];
Mno[1] = econi[1];
Do[Do[
  AA[j, i] = 0, {i, 26}], {j, 26}];
Do[Do[AA[j, i] = -Tr[econi[j].Mno[i]]/Tr[Mno[i].Mno[i]], {i, j - 1}];
  Mno[j] = econi[j] + Sum[AA[j, i]Mno[i], {i, j - 1}], {j, 2, 26}];
Do[Mnno[i] = -Sqrt[6]Mno[i]/Sqrt[Tr[Mno[i].Mno[i]]], {i, 26}];

```


vanishing terms, up to symmetries, which are not yet written in [1]:

$$\begin{aligned}
s_{1,54,55} &= \frac{1}{2}, & s_{1,56,58} &= -\frac{1}{2}, & s_{1,57,61} &= -\frac{1}{2}, & s_{1,59,60} &= -\frac{1}{2}, \\
s_{1,62,63} &= -\frac{1}{2}, & s_{1,64,66} &= \frac{1}{2}, & s_{1,65,69} &= \frac{1}{2}, & s_{1,67,68} &= \frac{1}{2}, \\
s_{1,71,72} &= -1, & s_{2,54,56} &= \frac{1}{2}, & s_{2,55,58} &= \frac{1}{2}, & s_{2,57,59} &= -\frac{1}{2}, \\
s_{2,60,61} &= -\frac{1}{2}, & s_{2,62,64} &= -\frac{1}{2}, & s_{2,63,66} &= -\frac{1}{2}, & s_{2,65,67} &= \frac{1}{2}, \\
s_{2,68,69} &= \frac{1}{2}, & s_{2,71,73} &= -1, & s_{3,54,58} &= \frac{1}{2}, & s_{3,55,56} &= -\frac{1}{2}, \\
s_{3,57,60} &= -\frac{1}{2}, & s_{3,59,61} &= \frac{1}{2}, & s_{3,62,66} &= \frac{1}{2}, & s_{3,63,64} &= -\frac{1}{2}, \\
s_{3,65,68} &= -\frac{1}{2}, & s_{3,67,69} &= \frac{1}{2}, & s_{3,72,73} &= -1, & s_{4,54,57} &= \frac{1}{2}, \\
s_{4,55,61} &= \frac{1}{2}, & s_{4,56,59} &= \frac{1}{2}, & s_{4,58,60} &= -\frac{1}{2}, & s_{4,62,65} &= -\frac{1}{2}, \\
s_{4,63,69} &= -\frac{1}{2}, & s_{4,64,67} &= -\frac{1}{2}, & s_{4,66,68} &= \frac{1}{2}, & s_{4,71,74} &= -1, \\
s_{5,54,61} &= \frac{1}{2}, & s_{5,55,57} &= -\frac{1}{2}, & s_{5,56,60} &= \frac{1}{2}, & s_{5,58,59} &= \frac{1}{2}, \\
s_{5,62,69} &= \frac{1}{2}, & s_{5,63,65} &= -\frac{1}{2}, & s_{5,64,68} &= \frac{1}{2}, & s_{5,66,67} &= \frac{1}{2}, \\
s_{5,72,74} &= -1, & s_{6,54,59} &= \frac{1}{2}, & s_{6,55,60} &= -\frac{1}{2}, & s_{6,56,57} &= -\frac{1}{2}, \\
s_{6,58,61} &= -\frac{1}{2}, & s_{6,62,67} &= \frac{1}{2}, & s_{6,63,68} &= -\frac{1}{2}, & s_{6,64,65} &= -\frac{1}{2}, \\
s_{6,66,69} &= -\frac{1}{2}, & s_{6,73,74} &= -1, & s_{7,54,58} &= \frac{1}{2}, & s_{7,55,56} &= -\frac{1}{2}, \\
s_{7,57,60} &= \frac{1}{2}, & s_{7,59,61} &= -\frac{1}{2}, & s_{7,62,66} &= -\frac{1}{2}, & s_{7,63,64} &= \frac{1}{2}, \\
s_{7,65,68} &= -\frac{1}{2}, & s_{7,67,69} &= \frac{1}{2}, & s_{7,71,75} &= -1, & s_{8,54,56} &= -\frac{1}{2}, \\
s_{8,55,58} &= -\frac{1}{2}, & s_{8,57,59} &= -\frac{1}{2}, & s_{8,60,61} &= -\frac{1}{2}, & s_{8,62,64} &= -\frac{1}{2}, \\
s_{8,63,66} &= -\frac{1}{2}, & s_{8,65,67} &= -\frac{1}{2}, & s_{8,68,69} &= -\frac{1}{2}, & s_{8,72,75} &= -1, \\
s_{9,54,55} &= \frac{1}{2}, & s_{9,56,58} &= -\frac{1}{2}, & s_{9,57,61} &= \frac{1}{2}, & s_{9,59,60} &= \frac{1}{2}, \\
s_{9,62,63} &= \frac{1}{2}, & s_{9,64,66} &= -\frac{1}{2}, & s_{9,65,69} &= \frac{1}{2}, & s_{9,67,68} &= \frac{1}{2}, \\
s_{9,73,75} &= -1, & s_{10,54,60} &= \frac{1}{2}, & s_{10,55,59} &= -\frac{1}{2}, & s_{10,56,61} &= -\frac{1}{2}, \\
s_{10,57,58} &= -\frac{1}{2}, & s_{10,62,68} &= \frac{1}{2}, & s_{10,63,67} &= \frac{1}{2}, & s_{10,64,69} &= -\frac{1}{2}, \\
s_{10,65,66} &= -\frac{1}{2}, & s_{10,74,75} &= -1, & s_{11,54,59} &= \frac{1}{2}, & s_{11,55,60} &= \frac{1}{2}, \\
s_{11,56,57} &= -\frac{1}{2}, & s_{11,58,61} &= \frac{1}{2}, & s_{11,62,67} &= -\frac{1}{2}, & s_{11,63,68} &= -\frac{1}{2}, \\
s_{11,64,65} &= \frac{1}{2}, & s_{11,66,69} &= -\frac{1}{2}, & s_{11,71,76} &= -1, & s_{12,54,60} &= \frac{1}{2}, \\
s_{12,55,59} &= -\frac{1}{2}, & s_{12,56,61} &= -\frac{1}{2}, & s_{12,57,58} &= \frac{1}{2}, & s_{12,62,68} &= \frac{1}{2}, \\
s_{12,63,67} &= -\frac{1}{2}, & s_{12,64,69} &= -\frac{1}{2}, & s_{12,65,66} &= \frac{1}{2}, & s_{12,72,76} &= -1, \\
s_{13,54,57} &= -\frac{1}{2}, & s_{13,55,61} &= \frac{1}{2}, & s_{13,56,59} &= -\frac{1}{2}, & s_{13,58,60} &= -\frac{1}{2}, \\
s_{13,62,65} &= -\frac{1}{2}, & s_{13,63,69} &= \frac{1}{2}, & s_{13,64,67} &= -\frac{1}{2}, & s_{13,66,68} &= \frac{1}{2}, \\
s_{13,73,76} &= -1, & s_{14,54,56} &= \frac{1}{2}, & s_{14,55,58} &= -\frac{1}{2}, & s_{14,57,59} &= -\frac{1}{2}, \\
s_{14,60,61} &= \frac{1}{2}, & s_{14,62,64} &= \frac{1}{2}, & s_{14,63,66} &= -\frac{1}{2}, & s_{14,65,67} &= -\frac{1}{2}, \\
s_{14,68,69} &= \frac{1}{2}, & s_{14,74,76} &= -1, & s_{15,54,61} &= \frac{1}{2}, & s_{15,55,57} &= \frac{1}{2}, \\
s_{15,56,60} &= \frac{1}{2}, & s_{15,58,59} &= -\frac{1}{2}, & s_{15,62,69} &= \frac{1}{2}, & s_{15,63,65} &= \frac{1}{2}, \\
s_{15,64,68} &= \frac{1}{2}, & s_{15,66,67} &= -\frac{1}{2}, & s_{15,75,76} &= -1, & s_{16,54,60} &= \frac{1}{2}, \\
s_{16,55,59} &= -\frac{1}{2}, & s_{16,56,61} &= \frac{1}{2}, & s_{16,57,58} &= -\frac{1}{2}, & s_{16,62,68} &= -\frac{1}{2}, \\
s_{16,63,67} &= \frac{1}{2}, & s_{16,64,69} &= -\frac{1}{2}, & s_{16,65,66} &= \frac{1}{2}, & s_{16,71,77} &= -1, \\
s_{17,54,59} &= -\frac{1}{2}, & s_{17,55,60} &= -\frac{1}{2}, & s_{17,56,57} &= -\frac{1}{2}, & s_{17,58,61} &= \frac{1}{2}, \\
s_{17,62,67} &= -\frac{1}{2}, & s_{17,63,68} &= -\frac{1}{2}, & s_{17,64,65} &= -\frac{1}{2}, & s_{17,66,69} &= \frac{1}{2}, \\
s_{17,72,77} &= -1, & s_{18,54,61} &= \frac{1}{2}, & s_{18,55,57} &= \frac{1}{2}, & s_{18,56,60} &= -\frac{1}{2}, \\
s_{18,58,59} &= \frac{1}{2}, & s_{18,62,69} &= \frac{1}{2}, & s_{18,63,65} &= \frac{1}{2}, & s_{18,64,68} &= -\frac{1}{2}, \\
s_{18,66,67} &= \frac{1}{2}, & s_{18,73,77} &= -1, & s_{19,54,58} &= -\frac{1}{2}, & s_{19,55,56} &= -\frac{1}{2}, \\
s_{19,57,60} &= -\frac{1}{2}, & s_{19,59,61} &= -\frac{1}{2}, & s_{19,62,66} &= -\frac{1}{2}, & s_{19,63,64} &= -\frac{1}{2}, \\
s_{19,65,68} &= -\frac{1}{2}, & s_{19,67,69} &= -\frac{1}{2}, & s_{19,74,77} &= -1, & s_{20,54,57} &= \frac{1}{2},
\end{aligned}$$

$$\begin{aligned}
s_{20,55,61} &= -\frac{1}{2}, & s_{20,56,59} &= -\frac{1}{2}, & s_{20,58,60} &= -\frac{1}{2}, & s_{20,62,65} &= \frac{1}{2}, \\
s_{20,63,69} &= -\frac{1}{2}, & s_{20,64,67} &= -\frac{1}{2}, & s_{20,66,68} &= -\frac{1}{2}, & s_{20,75,77} &= -\frac{1}{2}, \\
s_{21,54,55} &= \frac{1}{2}, & s_{21,56,58} &= \frac{1}{2}, & s_{21,57,61} &= \frac{1}{2}, & s_{21,59,60} &= -\frac{1}{2}, \\
s_{21,62,63} &= \frac{1}{2}, & s_{21,64,66} &= \frac{1}{2}, & s_{21,65,69} &= \frac{1}{2}, & s_{21,67,68} &= -\frac{1}{2}, \\
s_{21,76,77} &= -\frac{1}{2}, & s_{22,53,61} &= \frac{1}{2}, & s_{22,61,70} &= -\frac{\sqrt{3}}{2}, & s_{22,62,78} &= -\frac{1}{2}, \\
s_{22,63,74} &= -\frac{1}{2}, & s_{22,64,77} &= -\frac{1}{2}, & s_{22,65,72} &= \frac{1}{2}, & s_{22,66,76} &= -\frac{1}{2}, \\
s_{22,67,75} &= \frac{1}{2}, & s_{22,68,73} &= \frac{1}{2}, & s_{22,69,71} &= \frac{1}{2}, & s_{23,53,57} &= -\frac{1}{2}, \\
s_{23,57,70} &= -\frac{\sqrt{3}}{2}, & s_{23,62,74} &= \frac{1}{2}, & s_{23,63,78} &= -\frac{1}{2}, & s_{23,64,76} &= -\frac{1}{2}, \\
s_{23,65,71} &= -\frac{1}{2}, & s_{23,66,77} &= \frac{1}{2}, & s_{23,67,73} &= \frac{1}{2}, & s_{23,68,75} &= -\frac{1}{2}, \\
s_{23,69,72} &= \frac{1}{2}, & s_{24,53,60} &= -\frac{1}{2}, & s_{24,60,70} &= -\frac{\sqrt{3}}{2}, & s_{24,62,77} &= \frac{1}{2}, \\
s_{24,63,76} &= \frac{1}{2}, & s_{24,64,78} &= -\frac{1}{2}, & s_{24,65,75} &= \frac{1}{2}, & s_{24,66,74} &= -\frac{1}{2}, \\
s_{24,67,72} &= -\frac{1}{2}, & s_{24,68,71} &= -\frac{1}{2}, & s_{24,69,73} &= \frac{1}{2}, & s_{25,53,55} &= \frac{1}{2}, \\
s_{25,55,70} &= \frac{\sqrt{3}}{2}, & s_{25,62,72} &= -\frac{1}{2}, & s_{25,63,71} &= \frac{1}{2}, & s_{25,64,75} &= -\frac{1}{2}, \\
s_{25,65,78} &= -\frac{1}{2}, & s_{25,66,73} &= \frac{1}{2}, & s_{25,67,77} &= -\frac{1}{2}, & s_{25,68,76} &= \frac{1}{2}, \\
s_{25,68,74} &= \frac{1}{2}, & s_{26,53,59} &= -\frac{1}{2}, & s_{26,59,70} &= -\frac{\sqrt{3}}{2}, & s_{26,62,76} &= \frac{1}{2}, \\
s_{26,63,77} &= -\frac{1}{2}, & s_{26,64,74} &= \frac{1}{2}, & s_{26,65,73} &= -\frac{1}{2}, & s_{26,66,78} &= -\frac{1}{2}, \\
s_{26,67,71} &= -\frac{1}{2}, & s_{26,68,72} &= \frac{1}{2}, & s_{26,69,75} &= \frac{1}{2}, & s_{27,53,58} &= \frac{1}{2}, \\
s_{27,58,70} &= \frac{\sqrt{3}}{2}, & s_{27,62,75} &= -\frac{1}{2}, & s_{27,63,73} &= -\frac{1}{2}, & s_{27,64,72} &= \frac{1}{2}, \\
s_{27,64,77} &= \frac{1}{2}, & s_{27,66,71} &= \frac{1}{2}, & s_{27,67,78} &= -\frac{1}{2}, & s_{27,68,74} &= -\frac{1}{2}, \\
s_{27,69,76} &= \frac{1}{2}, & s_{28,53,56} &= \frac{1}{2}, & s_{28,56,70} &= \frac{\sqrt{3}}{2}, & s_{28,62,73} &= -\frac{1}{2}, \\
s_{28,63,75} &= \frac{1}{2}, & s_{28,64,71} &= \frac{1}{2}, & s_{28,65,76} &= -\frac{1}{2}, & s_{28,66,72} &= -\frac{1}{2}, \\
s_{28,67,74} &= \frac{1}{2}, & s_{28,68,78} &= -\frac{1}{2}, & s_{28,69,77} &= \frac{1}{2}, & s_{29,53,54} &= -\frac{1}{2}, \\
s_{29,54,70} &= -\frac{\sqrt{3}}{2}, & s_{29,62,71} &= -\frac{1}{2}, & s_{29,63,72} &= -\frac{1}{2}, & s_{29,64,73} &= -\frac{1}{2}, \\
s_{29,65,74} &= -\frac{1}{2}, & s_{29,66,75} &= -\frac{1}{2}, & s_{29,67,76} &= -\frac{1}{2}, & s_{29,68,77} &= -\frac{1}{2}, \\
s_{29,69,78} &= -\frac{1}{2}, & s_{30,54,61} &= \frac{1}{2}, & s_{30,55,57} &= -\frac{1}{2}, & s_{30,56,60} &= -\frac{1}{2}, \\
s_{30,58,59} &= -\frac{1}{2}, & s_{30,62,69} &= -\frac{1}{2}, & s_{30,63,65} &= \frac{1}{2}, & s_{30,64,68} &= \frac{1}{2}, \\
s_{30,66,67} &= \frac{1}{2}, & s_{30,71,78} &= -\frac{1}{2}, & s_{31,54,57} &= -\frac{1}{2}, & s_{31,55,61} &= -\frac{1}{2}, \\
s_{31,56,59} &= \frac{1}{2}, & s_{31,58,60} &= -\frac{1}{2}, & s_{31,62,65} &= -\frac{1}{2}, & s_{31,63,69} &= -\frac{1}{2}, \\
s_{31,64,67} &= \frac{1}{2}, & s_{31,66,68} &= -\frac{1}{2}, & s_{31,72,78} &= -\frac{1}{2}, & s_{32,54,60} &= -\frac{1}{2}, \\
s_{32,55,59} &= -\frac{1}{2}, & s_{32,56,61} &= -\frac{1}{2}, & s_{32,57,58} &= -\frac{1}{2}, & s_{32,62,68} &= -\frac{1}{2}, \\
s_{32,63,67} &= -\frac{1}{2}, & s_{32,64,69} &= -\frac{1}{2}, & s_{32,65,66} &= -\frac{1}{2}, & s_{32,73,78} &= -\frac{1}{2}, \\
s_{33,54,55} &= \frac{1}{2}, & s_{33,56,58} &= \frac{1}{2}, & s_{33,57,61} &= -\frac{1}{2}, & s_{33,59,60} &= \frac{1}{2}, \\
s_{33,62,63} &= \frac{1}{2}, & s_{33,64,66} &= \frac{1}{2}, & s_{33,65,69} &= -\frac{1}{2}, & s_{33,67,68} &= \frac{1}{2}, \\
s_{33,74,78} &= -\frac{1}{2}, & s_{34,54,59} &= -\frac{1}{2}, & s_{34,55,60} &= \frac{1}{2}, & s_{34,56,57} &= -\frac{1}{2}, \\
s_{34,58,61} &= -\frac{1}{2}, & s_{34,62,67} &= -\frac{1}{2}, & s_{34,63,68} &= \frac{1}{2}, & s_{34,64,65} &= -\frac{1}{2}, \\
s_{34,66,69} &= -\frac{1}{2}, & s_{34,75,78} &= -\frac{1}{2}, & s_{35,54,58} &= \frac{1}{2}, & s_{35,55,56} &= \frac{1}{2}, \\
s_{35,57,60} &= -\frac{1}{2}, & s_{35,59,61} &= -\frac{1}{2}, & s_{35,62,66} &= \frac{1}{2}, & s_{35,63,64} &= \frac{1}{2}, \\
s_{35,65,68} &= -\frac{1}{2}, & s_{35,67,69} &= -\frac{1}{2}, & s_{35,76,78} &= -\frac{1}{2}, & s_{36,54,56} &= \frac{1}{2}, \\
s_{36,55,58} &= -\frac{1}{2}, & s_{36,57,59} &= \frac{1}{2}, & s_{36,60,61} &= -\frac{1}{2}, & s_{36,62,64} &= \frac{1}{2}, \\
s_{36,63,66} &= -\frac{1}{2}, & s_{36,65,67} &= \frac{1}{2}, & s_{36,68,69} &= -\frac{1}{2}, & s_{36,77,78} &= -\frac{1}{2}, \\
s_{37,53,62} &= -\frac{1}{2}, & s_{37,54,71} &= -\frac{1}{2}, & s_{37,55,72} &= \frac{1}{2}, & s_{37,56,73} &= \frac{1}{2}, \\
s_{37,57,74} &= \frac{1}{2}, & s_{37,58,75} &= \frac{1}{2}, & s_{37,59,76} &= \frac{1}{2}, & s_{37,60,77} &= \frac{1}{2}, \\
s_{37,61,78} &= \frac{1}{2}, & s_{38,53,63} &= -\frac{1}{2}, & s_{38,54,72} &= -\frac{1}{2}, & s_{38,55,71} &= -\frac{1}{2}, \\
s_{38,56,75} &= -\frac{1}{2}, & s_{38,57,78} &= -\frac{1}{2}, & s_{38,58,73} &= \frac{1}{2}, & s_{38,59,77} &= -\frac{1}{2}, \\
s_{38,60,76} &= \frac{1}{2}, & s_{38,61,74} &= \frac{1}{2}, & s_{39,53,64} &= -\frac{1}{2}, & s_{39,54,73} &= -\frac{1}{2},
\end{aligned}$$

$$\begin{aligned}
s_{39,55,75} &= \frac{1}{2}, & s_{39,56,71} &= -\frac{1}{2}, & s_{39,57,76} &= -\frac{1}{2}, & s_{39,58,72} &= -\frac{1}{2}, \\
s_{39,59,74} &= \frac{1}{2}, & s_{39,60,78} &= -\frac{1}{2}, & s_{39,61,77} &= \frac{1}{2}, & s_{40,53,65} &= -1, \\
s_{40,54,74} &= -\frac{1}{2}, & s_{40,55,78} &= \frac{1}{2}, & s_{40,56,76} &= \frac{1}{2}, & s_{40,57,71} &= -\frac{1}{2}, \\
s_{40,58,77} &= -\frac{1}{2}, & s_{40,59,73} &= -\frac{1}{2}, & s_{40,60,75} &= \frac{1}{2}, & s_{40,61,72} &= -\frac{1}{2}, \\
s_{41,53,66} &= -1, & s_{41,54,75} &= -\frac{1}{2}, & s_{41,55,73} &= -\frac{1}{2}, & s_{41,56,72} &= \frac{1}{2}, \\
s_{41,57,77} &= \frac{1}{2}, & s_{41,58,71} &= -\frac{1}{2}, & s_{41,59,78} &= -\frac{1}{2}, & s_{41,60,74} &= -\frac{1}{2}, \\
s_{41,61,76} &= \frac{1}{2}, & s_{42,53,67} &= -1, & s_{42,54,76} &= -\frac{1}{2}, & s_{42,55,77} &= \frac{1}{2}, \\
s_{42,56,74} &= -\frac{1}{2}, & s_{42,57,73} &= \frac{1}{2}, & s_{42,58,78} &= \frac{1}{2}, & s_{42,59,71} &= -\frac{1}{2}, \\
s_{42,60,72} &= -\frac{1}{2}, & s_{42,61,75} &= -\frac{1}{2}, & s_{43,53,68} &= -1, & s_{43,54,77} &= -\frac{1}{2}, \\
s_{43,55,76} &= -\frac{1}{2}, & s_{43,56,78} &= \frac{1}{2}, & s_{43,57,75} &= -\frac{1}{2}, & s_{43,58,74} &= \frac{1}{2}, \\
s_{43,59,72} &= \frac{1}{2}, & s_{43,60,71} &= -\frac{1}{2}, & s_{43,61,73} &= -\frac{1}{2}, & s_{44,53,69} &= -1, \\
s_{44,54,78} &= -\frac{1}{2}, & s_{44,55,74} &= -\frac{1}{2}, & s_{44,56,77} &= -\frac{1}{2}, & s_{44,57,72} &= \frac{1}{2}, \\
s_{44,58,76} &= -\frac{1}{2}, & s_{44,59,75} &= \frac{1}{2}, & s_{44,60,73} &= \frac{1}{2}, & s_{44,61,71} &= -\frac{1}{2}, \\
s_{45,53,71} &= -\frac{1}{2}, & s_{45,54,62} &= -\frac{1}{2}, & s_{45,55,63} &= -\frac{1}{2}, & s_{45,56,64} &= -\frac{1}{2}, \\
s_{45,57,65} &= -\frac{1}{2}, & s_{45,58,66} &= -\frac{1}{2}, & s_{45,59,67} &= -\frac{1}{2}, & s_{45,60,68} &= -\frac{1}{2}, \\
s_{45,61,69} &= -\frac{1}{2}, & s_{45,70,71} &= -\frac{\sqrt{3}}{2}, & s_{46,53,72} &= -\frac{1}{2}, & s_{46,54,63} &= -\frac{1}{2}, \\
s_{46,55,62} &= \frac{1}{2}, & s_{46,56,66} &= \frac{1}{2}, & s_{46,57,69} &= \frac{1}{2}, & s_{46,58,64} &= -\frac{1}{2}, \\
s_{46,59,68} &= \frac{1}{2}, & s_{46,60,67} &= -\frac{1}{2}, & s_{46,61,65} &= -\frac{1}{2}, & s_{46,70,72} &= -\frac{\sqrt{3}}{2}, \\
s_{47,53,73} &= -\frac{1}{2}, & s_{47,54,64} &= -\frac{1}{2}, & s_{47,55,66} &= -\frac{1}{2}, & s_{47,56,62} &= \frac{1}{2}, \\
s_{47,57,67} &= \frac{1}{2}, & s_{47,58,63} &= \frac{1}{2}, & s_{47,59,65} &= -\frac{1}{2}, & s_{47,60,69} &= \frac{1}{2}, \\
s_{47,61,68} &= -\frac{1}{2}, & s_{47,70,73} &= -\frac{\sqrt{3}}{2}, & s_{48,53,74} &= -\frac{1}{2}, & s_{48,54,65} &= -\frac{1}{2}, \\
s_{48,55,69} &= -\frac{1}{2}, & s_{48,56,67} &= -\frac{1}{2}, & s_{48,57,62} &= -\frac{1}{2}, & s_{48,58,68} &= \frac{1}{2}, \\
s_{48,59,64} &= \frac{1}{2}, & s_{48,60,66} &= -\frac{1}{2}, & s_{48,61,63} &= \frac{1}{2}, & s_{48,70,74} &= -\frac{\sqrt{3}}{2}, \\
s_{49,53,75} &= -\frac{1}{2}, & s_{49,54,66} &= -\frac{1}{2}, & s_{49,55,64} &= \frac{1}{2}, & s_{49,56,63} &= -\frac{1}{2}, \\
s_{49,57,68} &= -\frac{1}{2}, & s_{49,58,62} &= \frac{1}{2}, & s_{49,59,69} &= \frac{1}{2}, & s_{49,60,65} &= \frac{1}{2}, \\
s_{49,61,67} &= -\frac{1}{2}, & s_{49,70,75} &= -\frac{\sqrt{3}}{2}, & s_{50,53,76} &= -\frac{1}{2}, & s_{50,54,67} &= -\frac{1}{2}, \\
s_{50,55,68} &= -\frac{1}{2}, & s_{50,56,65} &= \frac{1}{2}, & s_{50,57,64} &= -\frac{1}{2}, & s_{50,58,69} &= -\frac{1}{2}, \\
s_{50,59,62} &= \frac{1}{2}, & s_{50,60,63} &= \frac{1}{2}, & s_{50,61,66} &= \frac{1}{2}, & s_{50,70,76} &= -\frac{\sqrt{3}}{2}, \\
s_{51,53,77} &= -\frac{1}{2}, & s_{51,54,68} &= -\frac{1}{2}, & s_{51,55,67} &= \frac{1}{2}, & s_{51,56,69} &= -\frac{1}{2}, \\
s_{51,57,66} &= \frac{1}{2}, & s_{51,58,65} &= -\frac{1}{2}, & s_{51,59,63} &= -\frac{1}{2}, & s_{51,60,62} &= \frac{1}{2}, \\
s_{51,61,64} &= \frac{1}{2}, & s_{51,70,77} &= -\frac{\sqrt{3}}{2}, & s_{52,53,78} &= -\frac{1}{2}, & s_{52,54,69} &= -\frac{1}{2}, \\
s_{52,55,65} &= \frac{1}{2}, & s_{52,56,68} &= -\frac{1}{2}, & s_{52,57,63} &= -\frac{1}{2}, & s_{52,58,67} &= \frac{1}{2}, \\
s_{52,59,66} &= -\frac{1}{2}, & s_{52,60,64} &= -\frac{1}{2}, & s_{52,61,62} &= \frac{1}{2}, & s_{52,70,78} &= -\frac{\sqrt{3}}{2}.
\end{aligned}$$

[illegible]

[illegible]

[illegible]

[illegible]

(C.9)

(C.10)

[illegible]

[illegible]

[illegible]

[illegible]

(C.17)

(C.18)

where $\kappa = \frac{i}{\sqrt{3}}$,

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

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